Machine Learning

SEU — 2027 Spring

Assignment 4

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§ Support Vector Machines and Kernel Methods §

Problem 1: Linearly Separable SVM

(1) Given a training set $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ with $y_i \in \{-1, +1\}$, write the primal optimization problem for the hard-margin support vector machine.

Proof. The hard-margin SVM primal problem is

$$\min_{\mathbf{w},b} \ \frac{1}{2}\|\mathbf{w}\|^2 \quad \text{s.t.} \quad y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1, \ \forall i, \tag{1.1}$$

which correctly classifies every training point while maximizing the geometric margin $2/\|\mathbf{w}\|$. □

(2) Write the corresponding Lagrangian dual problem.

Proof. Introducing multipliers $\alpha_i \geq 0$, the dual problem is

$$\max_{\boldsymbol{\alpha}} \ \sum_{i=1}^n \alpha_i - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^\top \mathbf{x}_j \quad \text{s.t.} \quad \sum_{i=1}^n \alpha_i y_i = 0, \ \alpha_i \geq 0. \tag{1.2}$$

The optimum satisfies $\mathbf{w}^* = \sum_i \alpha_i^* y_i \mathbf{x}_i$, with support vectors corresponding to $\alpha_i^* > 0$. □

Problem 2: Kernel Methods

(1) Prove that the Gaussian kernel $K(\mathbf{x}, \mathbf{z}) = \exp(-\|\mathbf{x}-\mathbf{z}\|^2/2\sigma^2)$ is a valid kernel (its Gram matrix is positive semi-definite).

Proof. Expand the Gaussian kernel as

$$K(\mathbf{x}, \mathbf{z}) = \exp\left(-\frac{\|\mathbf{x}\|^2}{2\sigma^2}\right) \exp\left(\frac{\mathbf{x}^\top \mathbf{z}}{\sigma^2}\right) \exp\left(-\frac{\|\mathbf{z}\|^2}{2\sigma^2}\right). \tag{2.1}$$

The middle factor $\exp(\mathbf{x}^\top \mathbf{z}/\sigma^2)$ expands as an infinite series of polynomial kernels, each term of which is positive semi-definite; nonnegative combinations and pointwise (Schur) products of PSD kernels remain PSD, and the outer positive scalar factors preserve PSD-ness, so K is a valid kernel. □

This assignment is due 20:00, January 8, 2027 and the date of submission is August 27, 2026.

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