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15-251 – Homework 5

Problem 1

Claim. The set of rational numbers ($\mathbb{Q}$) is countable.

Proof. Let $x \in \mathbb{Q}$. By definition we can write $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$. This uniquely determines a pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, so there is an injection $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z}$ is countable, so is $\mathbb{Q}$. □

Problem 2

Let E be an event and let $1_E : \Omega \to \{0, 1\}$ be its indicator random variable.

Claim. $\mathbb{E}[1_E] = \mathbb{P}[E]$.

Proof.

$$\begin{aligned}\mathbb{E}[1_E] &= \mathbb{P}[1_E = 1] \cdot 1 + \mathbb{P}[1_E = 0] \cdot 0 \\ &= \mathbb{P}[1_E = 1] \\ &= \mathbb{P}[E].\end{aligned}$$

□

Problem 3

Claim. $\mathsf{P} \neq \mathsf{NP}$ would follow from exhibiting a language in NP that is provably outside P; no such proof is currently known.

Proof. We have already shown $\mathsf{P} \subseteq \mathsf{NP}$, so it suffices to show $\mathsf{NP} \not\subseteq \mathsf{P}$. Fix a language $L \in \mathsf{NP}$.

TO-DO

□

Problem 4

Claim. For n shuffled cards compared against an identical ordered deck, the probability of exactly k matches is

$$\pi_k = \frac{1}{k!}\left(1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}\right).$$

Proof. First fix and order the k cards to be matched; the probability their positions agree is $\frac{1}{k!}$. For the remaining $n-k$ cards, we want none of them to match, i.e. the complement of $\bigcup_{i=1}^{n-k} A_i$ where A_i is the event that position i matches. By inclusion-exclusion,

$$\mathbb{P}\left[\bigcup_{i=1}^{n-k} A_i\right] = \sum_{i=1}^{n-k} \frac{(-1)^i}{i!},$$

so the probability of no further matches is $1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}$. Multiplying the two independent parts together gives π_k as claimed. □