TMA: M208: Pure Mathematics-03

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27 August 2026

Q 1

(i) Prove, by induction on n, that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$.

Base case. For $n = 1$, the left side is 1 and the right side is $\frac{1 \cdot 2 \cdot 3}{6} = 1$, so the statement holds.

Inductive step. Assume the identity holds for some $n \geq 1$. Adding the next term,

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6},$$

which is exactly the claimed formula evaluated at $n+1$. By induction the identity holds for all $n \geq 1$. ☻

(ii) Use the formula above to evaluate $\sum_{k=1}^{10} k^2$.

Substituting $n = 10$ gives $\frac{10 \cdot 11 \cdot 21}{6} = 385$.

Q 2

Solve the second-order linear differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

The auxiliary equation is $\lambda^2 - 5\lambda + 6 = 0$, which factorises to $(\lambda - 2)(\lambda - 3) = 0$, giving roots $\lambda = 2$ and $\lambda = 3$. Since these are real and distinct, the general solution is

$$y = A\mathrm{e}^{2x} + B\mathrm{e}^{3x}.$$

Q 4

(a) Find the eigenvalues of the matrix $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

The characteristic equation is $(4-\lambda)(3-\lambda) - 2 = 0$, i.e. $\lambda^2 - 7\lambda + 10 = 0$, which factorises to $(\lambda - 2)(\lambda - 5) = 0$. The eigenvalues are $\lambda = 2$ and $\lambda = 5$.

(b) Find an eigenvector corresponding to the smaller eigenvalue.

For $\lambda = 2$: $(4-2)v_1 + v_2 = 0 \Rightarrow v_2 = -2v_1$, so an eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.