Ada Lovelace
Fall 2027

Linear Algebra Homework 4

Problem 1. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

Solution. ($\Rightarrow$) Suppose A is invertible. If 0 were an eigenvalue, there would exist $v \neq 0$ with $Av = 0v = 0$, so $v \in \ker A$, contradicting that A is injective (invertible matrices have trivial kernel).

($\Leftarrow$) Suppose 0 is not an eigenvalue. Then $\ker A = \{0\}$, since $Av = 0$ would otherwise give an eigenvector for eigenvalue 0. A square matrix with trivial kernel is injective, hence bijective, hence invertible. $\square$

Problem 2. Determine, with justification, whether the following matrix is diagonalizable:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solution. The characteristic polynomial is $\det(A - \lambda I) = (2 - \lambda)^2$, so $\lambda = 2$ is the only eigenvalue, with algebraic multiplicity 2. The eigenspace is

$$\ker(A - 2I) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\{(1, 0)^\top\},$$

which has dimension 1. Since the geometric multiplicity (1) is strictly less than the algebraic multiplicity (2), A is **not** diagonalizable. $\square$

Problem 3. Let V be a finite-dimensional vector space and $T : V \to V$ a linear map. Show the following are equivalent.

(a) T is injective.

(b) T is surjective.

(c) T is bijective.

Solution. By the rank–nullity theorem, $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$.

(a) $\Rightarrow$ (b): If T is injective, $\dim(\ker T) = 0$, so $\dim(\operatorname{im} T) = \dim V$, and since $\operatorname{im} T \subseteq V$ with equal dimension, $\operatorname{im} T = V$, i.e. T is surjective.

(b) $\Rightarrow$ (c): If T is surjective, $\dim(\operatorname{im} T) = \dim V$, so $\dim(\ker T) = 0$, giving injectivity as well, hence bijectivity.

(c) $\Rightarrow$ (a): Immediate, since bijective implies injective. $\square$