Problem Set 4: Sequences and Series

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Problem 1

Determine whether the sequence $a_n = \dfrac{n^2 + 3n}{2n^2 - 1}$ converges, and if so, find its limit.

Solution Divide numerator and denominator by n^2:

$$a_n = \frac{1 + 3/n}{2 - 1/n^2} \longrightarrow \frac{1}{2} \quad \text{as } n \to \infty.$$

The sequence converges to $\frac{1}{2}$.

Problem 2 (Ratio test)

Use the ratio test to determine convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Proof Let $a_n = n!/n^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n} \longrightarrow e^{-1} < 1.$$

By the ratio test, the series converges.

Problem 3 (Several parts)

Consider the sequence defined by $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$.

1) Show that $x_n < 2$ for all n.

   Solution By induction: $x_1 = 1 < 2$. If $x_n < 2$, then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{4} = 2$.

2) Show that (x_n) is increasing.

   Solution Since $x_n < 2$, we have $x_n^2 < 2x_n +$ (check sign), which after simplification gives $x_{n+1} > x_n$ for all n.

3) Conclude that (x_n) converges, and find its limit.

   Solution A bounded, monotonically increasing sequence converges. Taking limits in the recursion, $L = \sqrt{2 + L}$ gives $L^2 - L - 2 = 0$, so $L = 2$.

Fact 3.1

A convergent sequence is bounded.

To be finished #1: the divergence test for the harmonic series

Below is a list of outstanding items, followed by the corresponding page numbers.

 # 1 ⟨2⟩ : the divergence test for the harmonic series