Problem Set 6 — Linear Algebra

Course: MATH 220

Due: October 9, 2026

Answer every question below. Full solutions are collected at the end of the sheet via `\printsolutions`, so students can be handed the questions alone and the answer key generated separately from the same source.

Exercise 1. 5 P.

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find the eigenvalues of A.

Exercise 2. 5 P.

For the same matrix A, find an eigenvector for each eigenvalue.

Exercise 3. 10 P.

Show that a 2×2 symmetric real matrix always has real eigenvalues.

Exercise 4. 10 P.

Let V be the vector space of polynomials of degree at most 2. Show that $\{1, x, x^2\}$ is a basis for V.

Answer Key

Solution 1.
The characteristic polynomial is $\det(A - \lambda I) = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$, which factors as $(\lambda - 1)(\lambda - 3)$. So the eigenvalues are $\lambda = 1$ and $\lambda = 3$.

Solution 2.
For $\lambda = 1$: $(A - I)v = 0$ gives $v = (1, -1)^\top$ (up to scale). For $\lambda = 3$: $(A - 3I)v = 0$ gives $v = (1, 1)^\top$ (up to scale).

Solution 3.
For $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ the characteristic polynomial is $\lambda^2 - (a + c)\lambda + (ac - b^2)$, with discriminant $(a + c)^2 - 4(ac - b^2) = (a - c)^2 + 4b^2 \geq 0$. A non-negative discriminant means both roots are real.

Solution 4.
Spanning: any $p(x) = a_0 + a_1 x + a_2 x^2$ with $a_i \in \mathbb{R}$ is by definition a linear combination of $1, x, x^2$.

Independence: if $a_0 + a_1 x + a_2 x^2 = 0$ for all x, comparing coefficients forces $a_0 = a_1 = a_2 = 0$. So the three polynomials are linearly independent, and together they form a basis.