Notes on Ordinary Differential Equations

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1 First-Order Linear Equations

A first-order linear ODE has the form $y' + p(x)y = q(x)$. Multiplying by the integrating factor $\mu(x) = e^{\int p(x)\,dx}$ turns the left side into an exact derivative:

$$\frac{d}{dx}\big[\mu(x)y\big] = \mu(x)q(x).$$

Remark This is the single most useful trick for first-order linear equations — everything else in this section is a variation on it.

Example 1.1 *Solve* $y' + 2xy = x$. *Here* $\mu(x) = e^{x^2}$, *so* $\left(e^{x^2}y\right)' = xe^{x^2}$, *giving* $e^{x^2}y = \frac{1}{2}e^{x^2} + C$, *i.e.* $y = \frac{1}{2} + Ce^{-x^2}$.

2 Second-Order Constant-Coefficient Equations

For $ay'' + by' + cy = 0$, the characteristic equation $a\lambda^2 + b\lambda + c = 0$ determines the solution's shape:

- two real roots $\lambda_1 \neq \lambda_2$: $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$;
- a repeated root λ: $y = (C_1 + C_2x)e^{\lambda x}$;

- complex roots $\alpha \pm i\beta$: $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Note The repeated-root case is exactly the boundary between oscillation and pure exponential decay/growth — it is worth sketching all three cases on the same axes to see why.

3 Numerical Methods

When no closed form exists, Euler's method approximates the solution of $y' = f(x, y)$, $y(x_0) = y_0$ by stepping $y_{n+1} = y_n + h f(x_n, y_n)$. It has local truncation error $O(h^2)$ and global error $O(h)$, per Butcher [1]. The classical fourth-order Runge–Kutta method improves this to global error $O(h^4)$ at the cost of four function evaluations per step, following Hairer, Nørsett, and Wanner [2].

References

[1] J. C. Butcher. *Numerical Methods for Ordinary Differential Equations*. 3rd ed. Wiley, 2016.

[2] Ernst Hairer, Syvert P. Nørsett, and Gerhard Wanner. *Solving Ordinary Differential Equations I: Nonstiff Problems*. 2nd ed. Springer, 1993.