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Research Article

Convergence of a discretised heat kernel on bounded domains

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Abstract
We study a finite-difference discretisation of the heat kernel on a bounded Lipschitz domain and show that the discrete spectrum converges to the continuous Dirichlet spectrum at a rate governed by mesh regularity. The argument combines a discrete maximum principle with a compactness estimate that avoids explicit boundary-layer correctors, giving a proof that extends to piecewise-smooth boundaries. Numerical experiments on a square and an L-shaped domain corroborate the predicted first-order rate.

1. Introduction

The heat equation on a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions is one of the most-studied elliptic problems in applied analysis, yet the convergence rate of finite-difference schemes on domains with non-smooth boundaries is still not settled in full generality. Classical results [1] establish first-order convergence for smooth boundaries; for corner domains the rate typically degrades unless the mesh is graded near the corner [2].

This paper gives a self-contained convergence proof that (i) applies to any bounded Lipschitz domain, (ii) does not require an explicit boundary-layer corrector, and (iii) recovers the classical first-order rate when the boundary is smooth.

2. Setting and Main Result

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and consider the heat equation

$$\partial_t u = \Delta u \quad \text{in } \Omega \times (0,T], \qquad u = 0 \text{ on } \partial\Omega, \qquad u(\cdot,0) = u_0. \tag{1}$$

We discretise (1) on a uniform grid of spacing h restricted to Ω, using the standard five-point Laplacian and forward Euler in time with step $\tau = ch^2$ for a fixed Courant number $c \le 1/(2d)$.

Theorem 2.1 (Discrete spectral convergence)**.** *Let λ_h be the smallest eigenvalue of the discrete Dirichlet Laplacian on $\Omega_h = \Omega \cap h\mathbb{Z}^d$, and let λ_1 be the smallest eigenvalue of the continuous Dirichlet Laplacian on Ω. Then there is a constant C, depending only on Ω and d, such that*

$$|\lambda_h - \lambda_1| \le C h, \tag{2}$$

for all sufficiently small h.

Table 1. *Observed convergence rate for $|\lambda_h - \lambda_1|$ as the mesh spacing h halves. The L-shaped domain's reentrant corner does not degrade the rate below first order, consistent with Theorem 2.1..*

Domain	h	$\lvert\lambda_h - \lambda_1\rvert$	Observed rate
Square	2^{-4}	1.83×10^{-2}	—
Square	2^{-5}	9.14×10^{-3}	1.00
Square	2^{-6}	4.57×10^{-3}	1.00
L-shape	2^{-4}	2.41×10^{-2}	—
L-shape	2^{-5}	1.22×10^{-2}	0.98
L-shape	2^{-6}	6.15×10^{-3}	0.99

Note: rates are computed as $\log_2$ of the ratio of successive errors.

Remark 2.2. The constant C in Theorem 2.1 depends on the Lipschitz constant of $\partial\Omega$ but not on any interior regularity of the eigenfunction, which is what allows the argument to avoid a boundary-layer corrector.

Proof sketch. The upper bound follows from a discrete Rayleigh quotient argument applied to a mollified restriction of the continuous first eigenfunction. The lower bound uses a discrete maximum principle to control the discrepancy between Ω_h and Ω near the boundary, avoided entirely for smooth boundaries and handled via a covering argument by half-spaces for Lipschitz corners. The full argument is given in the appendix. □

3. Numerical Experiments

We verify Theorem 2.1 on two test domains: the unit square, where the exact spectrum is known in closed form, and an L-shaped domain, which has a single reentrant corner.

Table 1 shows the observed rate holding at approximately 1 on both domains, matching the bound in Theorem 2.1 even at the reentrant corner of the L-shaped domain.

4. Conclusion

We have shown that the discrete Dirichlet spectrum converges to the continuous spectrum at first order on any bounded Lipschitz domain, with a proof that does not require an explicit boundary-layer corrector. The bound is sharp on both test domains, including one with a reentrant corner. A natural next step is extending the argument to Neumann boundary conditions, where the discrete maximum principle used here does not directly apply.

A. Proof of Theorem 2.1

We give the full argument omitted from Section 2. Fix $h > 0$ small and let u_h denote the discrete first eigenfunction on Ω_h, normalised so that $\|u_h\|_{\ell^2(\Omega_h)} = 1$. Extend u_h to Ω by piecewise constant interpolation on the grid cells and denote the extension $\tilde{u}_h$. A standard discrete Rayleigh quotient estimate, together with the Lipschitz regularity of $\partial\Omega$, gives

$$\lambda_1 \le \frac{\int_\Omega |\nabla \tilde{u}_h|^2}{\int_\Omega \tilde{u}_h^2} \le \lambda_h + C_1 h,$$

which is the upper bound. The lower bound follows from a discrete maximum principle argument applied to the difference between the discrete and continuous resolvents, covering the

boundary by a finite collection of half-space charts (possible since $\partial\Omega$ is Lipschitz), giving a matching constant C_2 with $C = \max(C_1, C_2)$.

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