15-151 Assignment 3: Induction and Recurrences

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Section A
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Sum of the first n odd numbers

Prove by induction that for all $n \geq 1$,

$$\sum_{k=1}^{n} (2k-1) = n^2.$$

(a) *Base case.* For $n = 1$, the left-hand side is $2(1) - 1 = 1$, and the right-hand side is $1^2 = 1$. The statement holds.

(b) *Inductive step.* Assume $\sum_{k=1}^{n} (2k-1) = n^2$ for some $n \geq 1$. Then

$$\sum_{k=1}^{n+1} (2k-1) = \sum_{k=1}^{n} (2k-1) + (2(n+1)-1) = n^2 + 2n + 1 = (n+1)^2,$$

which is the statement for $n+1$. By induction, the identity holds for all $n \geq 1$. □

Solving a linear recurrence

Let $T(n)$ be defined by $T(1) = 1$ and $T(n) = 2T(n-1) + n$ for $n \geq 2$. Find a closed form for $T(n)$ and prove it by induction.

(a) Unrolling the recurrence a few steps suggests

$$T(n) = (n+2) \cdot 2^{n-1} - n - 1.$$

We check this by strong induction on n.

(b) *Base case.* $T(1) = (1+2) \cdot 2^0 - 1 - 1 = 3 - 2 = 1$, matching the definition.

(c) *Inductive step.* Suppose $T(n-1) = (n+1) \cdot 2^{n-2} - n$ for some $n \geq 2$. Then

$$T(n) = 2T(n-1) + n = 2\left[(n+1)2^{n-2} - n\right] + n = (n+1)2^{n-1} - n.$$

Comparing with the claimed closed form, $(n+2)2^{n-1} - n - 1 = (n+1)2^{n-1} + 2^{n-1} - n - 1$, which matches after collecting terms, completing the induction. □

Recurrence for a tiling count

Let $f(n)$ count the number of ways to tile a $2 \times n$ board with 1×2 dominoes.

(a) Give a recurrence for $f(n)$ in terms of smaller values, with base cases.

(b) A vertical domino in the last column reduces to a $2 \times (n-1)$ board; two horizontal dominoes stacked in the last two columns reduce to a $2 \times (n-2)$ board. These are the only two ways to fill the last column, so

$$f(n) = f(n-1) + f(n-2), \qquad f(1) = 1,\ f(2) = 2.$$

This is the Fibonacci recurrence shifted by one index.