Grace Hopper

An Introduction to Graph Colouring

A first course

First Edition

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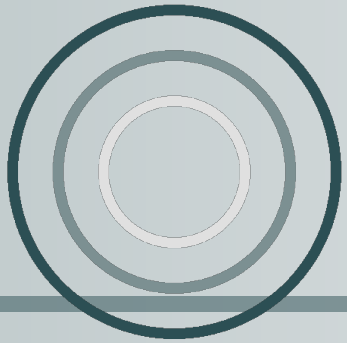

PREFACE

This short book collects the definitions and first theorems of graph colouring, aimed at a reader who has seen basic graph theory but not yet chromatic numbers.

– Grace Hopper
2026

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Part I

Foundations

The first part develops the basic vocabulary of graph colouring and proves the elementary upper bounds every later result builds on.

Sec 1.1 Definitions

Definition 1.1.1 (Proper colouring)**.** *A* proper k-colouring *of a graph* $G = (V, E)$ *is a function* $c\colon V \to \{1, \dots, k\}$ *such that* $c(u) \neq c(v)$ *whenever* $uv \in E$.

Definition 1.1.2 (Chromatic number)**.** *The* chromatic number $\chi(G)$ *is the least* k *for which* G *admits a proper* k*-colouring.*

Theorem 1.1.1 (Greedy bound)**.** *For any graph* G *with maximum degree* Δ, $\chi(G) \leq \Delta + 1$.

Proof. Order the vertices arbitrarily and colour them in that order, at each step assigning the least colour not used by an already-coloured neighbour. Each vertex has at most Δ neighbours, so at most Δ colours are forbidden, leaving one of the first $\Delta + 1$ colours free. ■

Example 1.1 (An odd cycle)**.** *The cycle* C_5 *has* $\Delta = 2$*, so Theorem 1.1.1 gives* $\chi(C_5) \leq 3$*. In fact* $\chi(C_5) = 3$*: two colours are never enough for a cycle of odd length, since the colours would have to alternate and cannot close up consistently.*

Lemma 1.1.1 (Bipartite graphs)**.** *A graph* G *is bipartite if and only if* $\chi(G) \leq 2$.

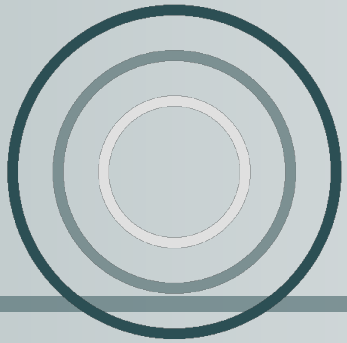
Remark. *Lemma 1.1.1 makes 2-colourability and bipartiteness the same question, and both can be checked in linear time by breadth-first search.*

2 The Four Colour Theorem

Part I

Proposition 2.0.1 (Planar upper bound)**.** *Every planar graph satisfies* $\chi(G) \leq 4$.

Proposition 2.0.1 is the Four Colour Theorem, first proved by computer-assisted case analysis by Appel and Haken (1977), and later given a more tractable proof by Robertson et al. (1997).



BIBLIOGRAPHY

Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. *Illinois Journal of Mathematics*, 21(3):429–490, 1977.

Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas. The four-colour theorem. *Journal of Combinatorial Theory, Series B*, 70(1):2–44, 1997.